Grade 6 Math Circles
February 7/8/9, 2023
Functions, Relations, and Graphing

What are Relations and Functions?

Relations

A **relation** describes the **relation**ship between two things. As an example, let's consider the relation between height and age. We can think of it as putting someone's age into a special machine and the machine will spit out their height. In this case, we call age the **input** and height the **output**.

Functions

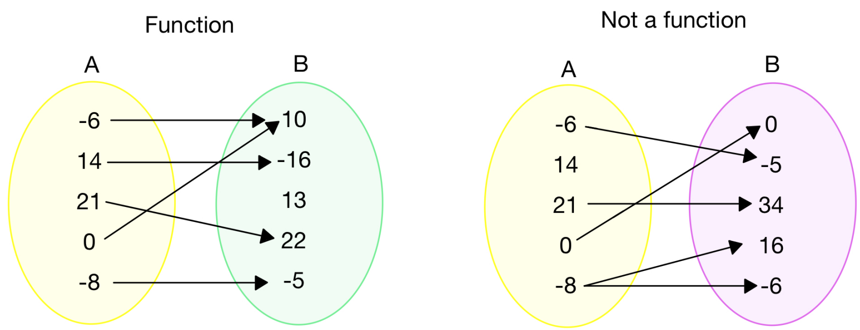
A **function** is a specific type of relation where each input can have only one output. The example above about height and age would not be considered a function. Consider a room full of 11-year-old students. Would they all be the same height? No, it is very unlikely for them to all be the exact same height. In fact, we know that the heights of all the 11-year-olds in the world vary greatly. Thus, it is possible for people of different heights to be the same age and thus one input (age) could lead to multiple outputs (height).

To get a better understanding of functions, let's imagine that we place a steaming hot cup of hot chocolate on a table at noon. At noon, can the hot chocolate be more than one temperature? What about at 4 pm? No, we know that this is impossible. This process follows our definition of a function since one input (the time) leads to one and only one output (the temperature). We can think of it like putting the time of day into a special machine and it spitting out the temperature of the hot chocolate.

Stop and Think

What are some other functions and relations we see in our every day lives?

There are so many functions and relations all around us but for this lesson, we will focus on the relationships between numbers.



The diagram above is a great visual representation of the differences between a function and a relation. Here, the A oval represents the input and the B ovals represent the output. The arrows going from the input to the output indicate which input value(s) goes to which output value(s). Can you see where the difference between these two diagrams is that makes the one on the left a function and the one on the right not a function?

Exercise 1

Classify each of the relations as a function or not a function.

a)

Input	Output
0	5
3	6
3	7
6	10

b)

Input	Output
-10	14
-5	0
0	6
2	2

c)

Input	Output
-5	4
0	7
3	14
10	4

This lesson will focus mainly on relations that are functions.

Notation

Now that we know what functions and relations are, how do we write them in math? First, we need to understand what a variable is.

A **variable** is a letter that we use to represent any number. We can think of it as a placeholder. For example, in $2a + 3$, we can put any number into the place where the a is. If we input $a = 5$, the expression will become $2(5) + 3 = 10 + 3 = 13$. It's important to remember that whenever we substitute a value for a variable into an expression, we need to put brackets around it, like we did in the example above with the 5.

Stop and Think

Why are brackets around the value of the variable so important?

When working with functions and relations, we often use the variable x to represent the input and the variable y to represent the output.

Functions and relations are usually of the form $y =$ (some expression involving x).

Example A

All of the following are examples of functions.

a) $y = x+12$ b) $y = 2x+1$ c) $y = 100$ d) $y = (x)(x)(x)+\frac{3}{4}$ e) $y = 3(x)(x)-4x+2$

Exercise 2

Find the outputs of the function $y = 3x - 2$ given the following inputs.

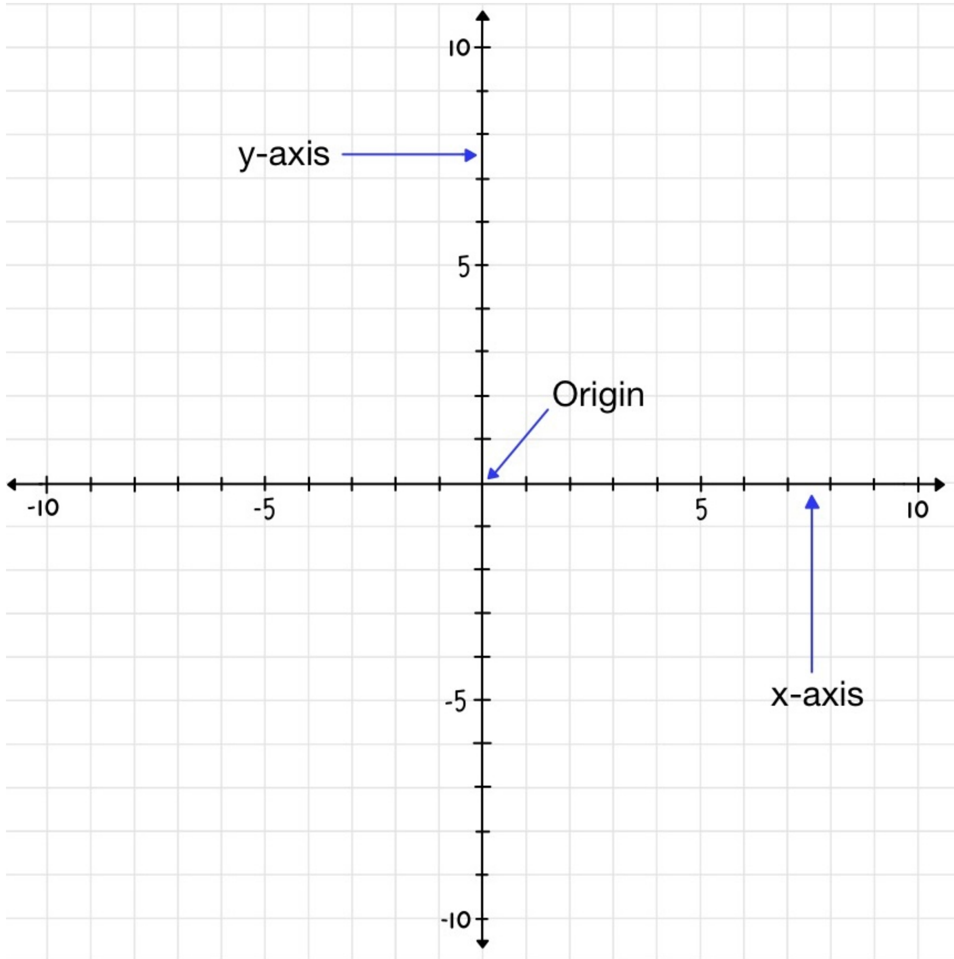
a) $x = 0$ b) $x = 1$ c) $x = -5$ d) $x = 10$ e) $x = -7$

Note: There are other ways of expressing functions but for this lesson we will use this way with x and y.

Graphing

The Cartesian Plane

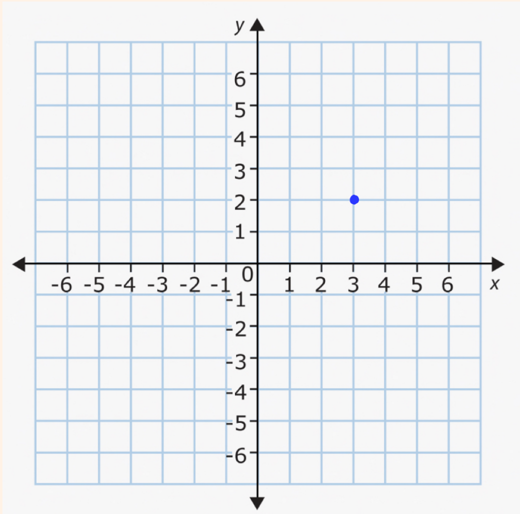
The Cartesian Plane is made up of two number lines: one running horizontally and the other running vertically. We call the horizontal number line the **x-axis** and the vertical number line is called the **y-axis**. The x-axis and the y-axis intersect where both of them are 0. We call this point of intersection the **origin**.



We can place points anywhere on the Cartesian Plane. Each point will have two numbers corresponding to it. The first one is called the x**-coordinate** and that number decides where along the x-axis that point will lay. Similarly, the second number is called the y**-coordinate** and it determines where along the y-axis the point will be. We write the coordinates of a point in brackets in the form (x-coordinate, y-coordinate).

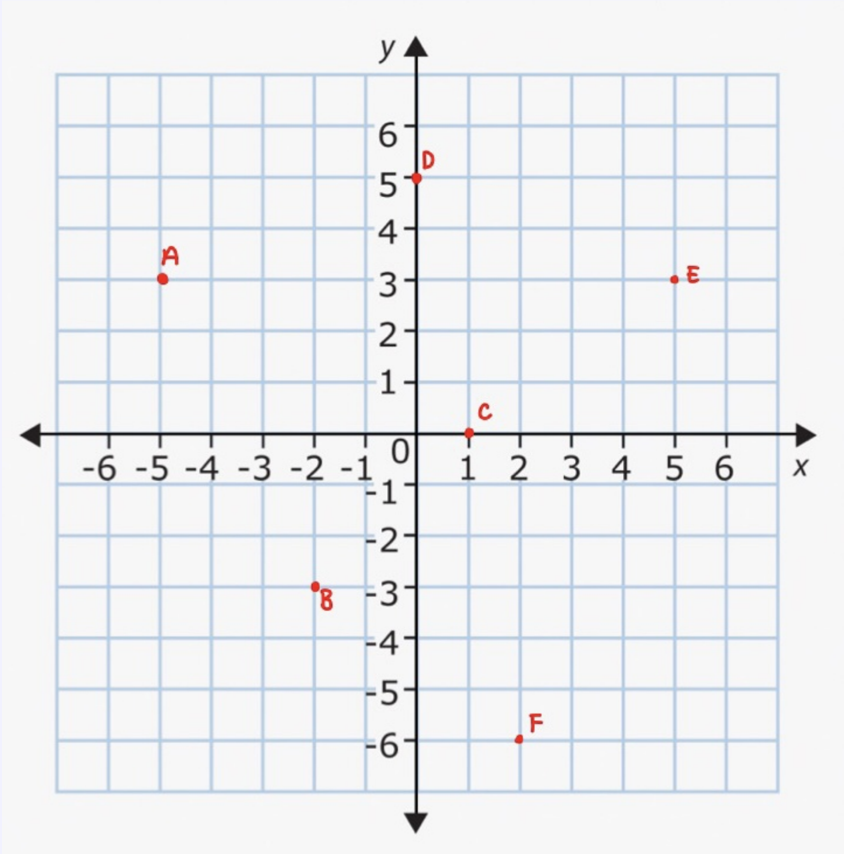
Example B

Place the point $(3, 2)$ on the Cartesian Plane.



Exercise 3

Write the coordinates for each of the points on the following Cartesian Plane:



Graphing Functions

We know the words input and output but how do they relate to graphing? Recall that we write functions in the form of $y =$ (some expression involving x) where the x is the input and y is the output. This translates nicely to the Cartesian plane where we can consider the x-coordinate of a point as the input and the y-coordinate as the output.

The simplest way to graph a function is to make a table with some x-coordinates and find the corresponding y-coordinates using the given function. Let's walk through an example together.

Example C

Sketch the function $y = 3x + 2$.

Start with a table with a variety of x-coordinates. Our goal is to use the function to find the corresponding y-coordinates and complete the table.

x	-2	-1	0	1	2	3
y						

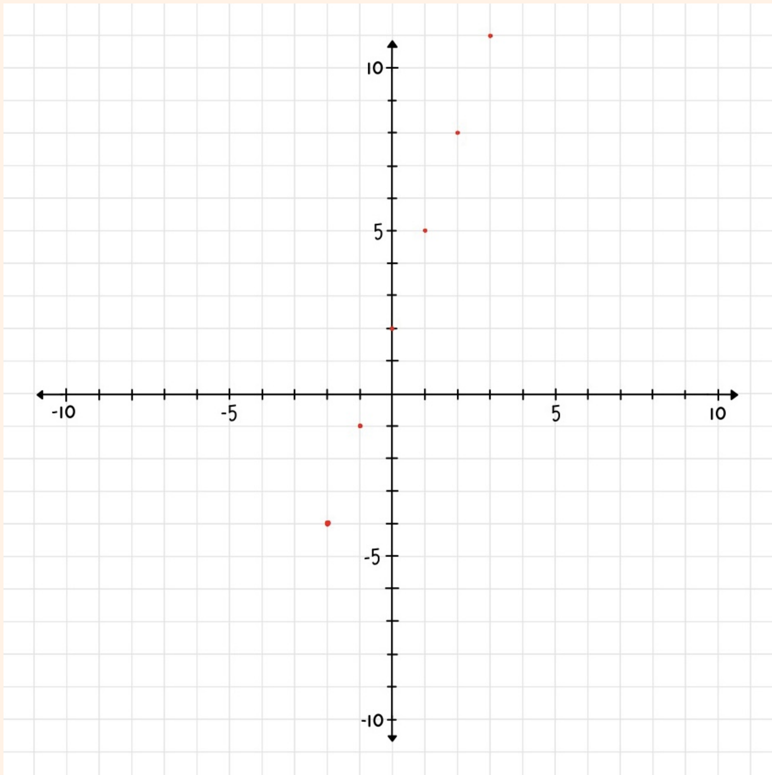
First, we will input $x = -2$ into our function $y = 3x + 2$. To do this, simply replace the x in the function with -2 in brackets as follows:

$$y = 3(-2) + 2 = -6 + 2 = -4$$

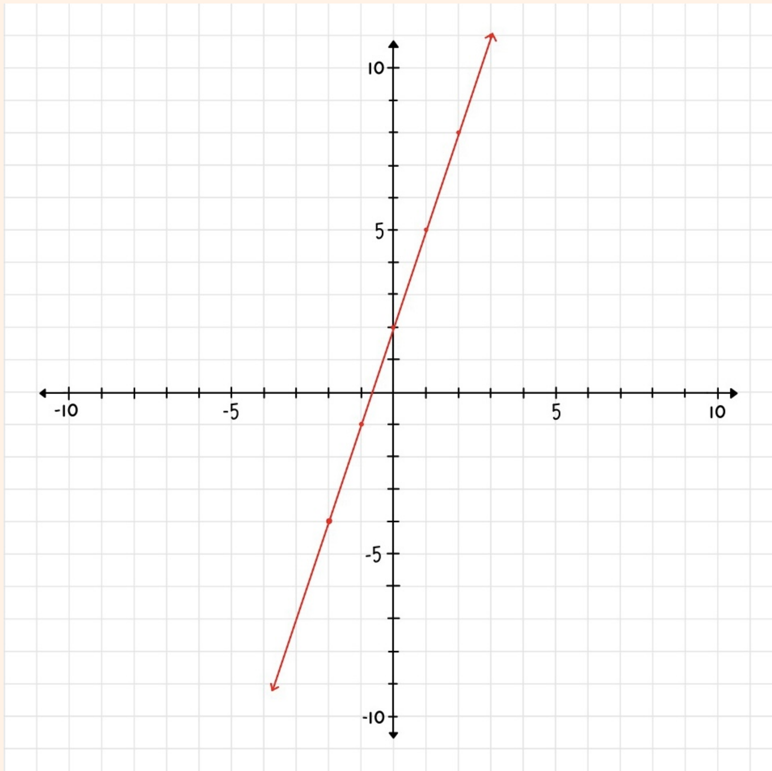
Thus, the y-coordinate that corresponds to $x = -2$ is -4. So we know the point $(-2, -4)$ is on our graph.

We can continue this method of substituting in x-coordinates to the function until we have a completed table. We should end up with the coordinates $(-2, -4)$, $(-1, -1)$, $(0, 2)$, $(1, 5)$, $(2, 8)$, $(3, 11)$.

Now that we have a few coordinates of points on the graph, we can plot these points on the Cartesian Plane:



It's pretty easy to notice that connecting the plot points will form a straight line. In fact, if we were to plot any more points that satisfy $y = 3x + 2$, they would all fall onto the same line. Since we now have a good idea of what the function looks like, we can connect these dots with a line. This resulting line is the graph of the function $y = 3x+2$ and we are done.



Note: To show that this line goes on forever, we add arrows to the end.

A Helpful Tip

When choosing which x-coordinates to input, it's always a good idea to have a mix of positive and negative numbers to see how the function behaves on both sides of the plane. Another good rule of thumb is to pick at least 5 points to work with so you have a good idea what the graph looks like before connecting the dots.

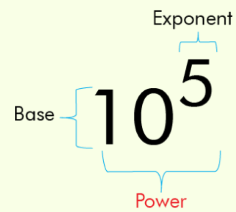
Stop and Think

Will choosing 5 points to plot always give a good idea of what the graph will look like?

A Quick Review of Exponents

Exponents are a simple way of writing repeated multiplication of the same number.

The **base** is the number we are repeatedly multiplying and the **exponent** tells us how many times to multiply the base. The base and the exponent together are called the **power**.



Here are a few examples of expanding exponents:

a) $2^4 = 2 \times 2 \times 2 \times 2$ b) $5^2 = 5 \times 5$ c) $x^3 = x \times x \times x$

Exercise 4

Create a table with 5 to 8 x-coordinates and find the corresponding y-coordinates for each function. Then, plot the points and connect the dots to find the graph.

a) $y = 5x - 3$ b) $y = x^2 + 4$ c) $y = 5$ d) $y = 3x^2$

Common Functions

Linear functions

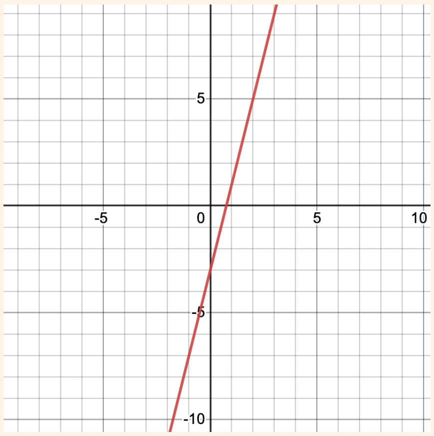
What was the shape of the function we sketched together in the example? It turned out to be a line. The function $y = 3x + 2$ is a special type of function conveniently called a **linear function**. They take the general form of

$$y = mx + b$$

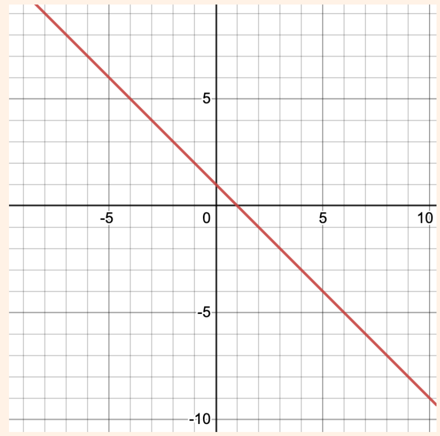
The value of m is the slope. The **slope** is what determines the steepness of the line. In our example $y = 3x + 2$, the slope of this function is 3 since 3 multiplies the x variable. What this means is that when the x-coordinate is increased by 1, the y-coordinate increases by 3 units.

The value of b is the y-intercept. The **y-intercept** is where the function intersects the y-axis. Looking at the sketch of our $y = 3x + 2$ example, we can see that the function intersects the y-axis at the point $(0, 2)$. Thus, 2 is the y-intercept of the function.

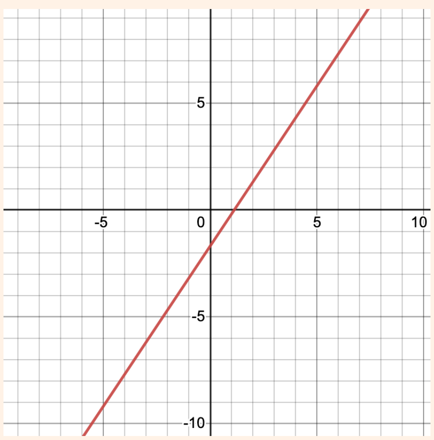
Example D



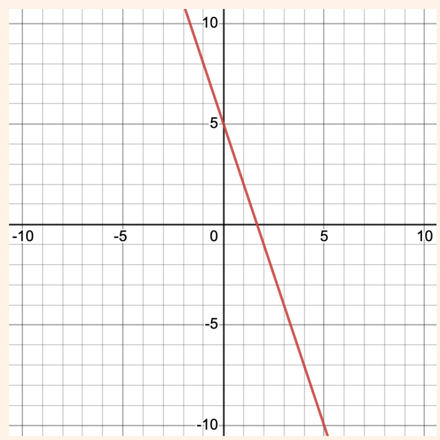
$y = 4x - 3$



$y = -x + 1$



$y = \frac{3}{2}x - \frac{5}{3}$



$y = -3x + 5$

Exercise 5

Find the slopes and y-intercepts in each of the examples above.

We notice in the above examples that a negative m-value results in the graph sloping down rather than up.

Stop and Think

Is the line steeper when the m value is closer or further from 0? What happens when $m = 0$?

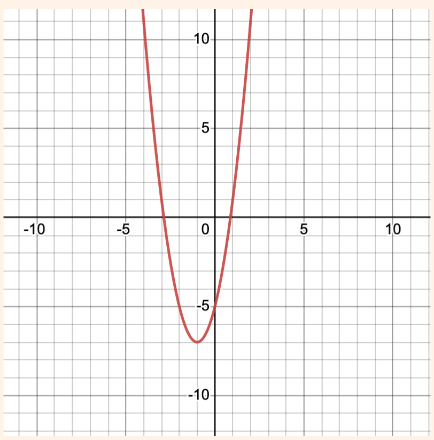
Quadratic Functions

Quadratic functions have a term containing x^2 and look U-shaped. This shape is called a **parabola**. Quadratic functions take the form:

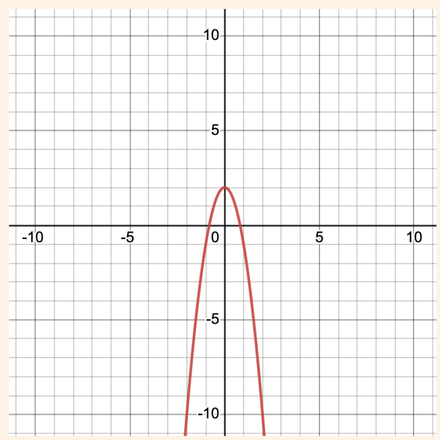
$$y = ax^2 + bx + c$$

In this form, the value of c is the y-intercept and the sign of a determines whether the parabola is pointing up or down. Quadratic functions do not necessarily have a slope but their width (how fat or skinny the parabola is) is determined by the a and b values.

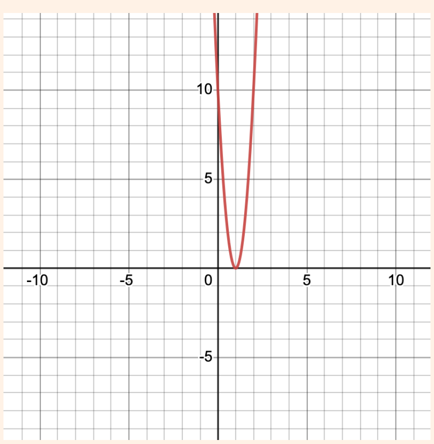
Example E



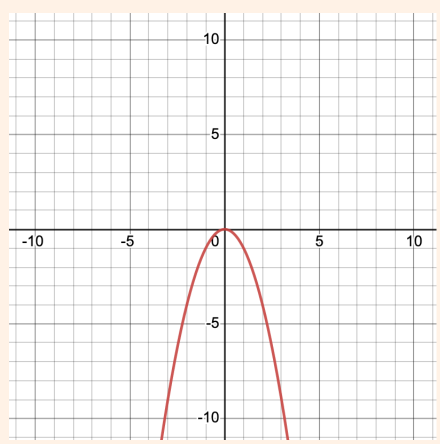
$y = 2x^2 + 4x - 5$



$y = -3x^2 + 2$



$y = 10x^2 - 20x + 10$



$y = -x^2$

The lowest point on a parabola that opens upwards is called the **vertex** (vertices is the plural form). The vertex on a parabola that opens downward is the highest point of the function.

Exercise 6

Find the vertices in each of the examples above.

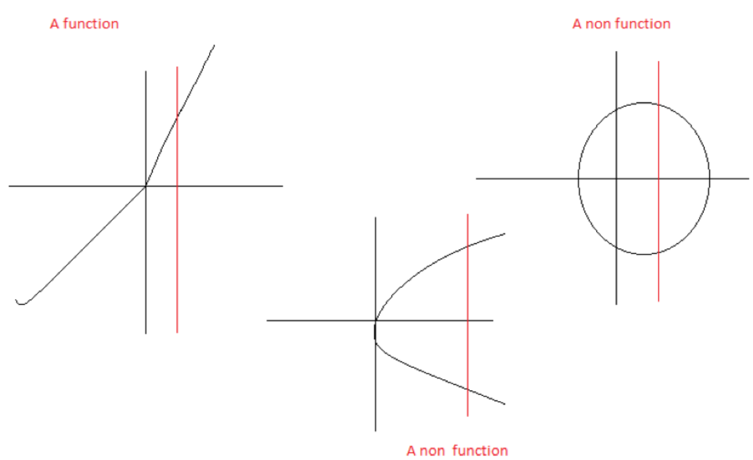
If you're interested in experimenting more with graphing, be sure to check out Desmos and Geogebra. These tools allow you to input any function and see how the graph looks. Also, be sure to come to next week's lesson where we'll learn more about graph theory!

Vertical Line Test

We can use graphing to tell if something is a function or relation using something called the **vertical line test**.

Given the graph of a relation, it is...

- a function if we can draw a vertical line anywhere on the graph and the graph intersects that line only once
- not a function if we can draw a vertical line somewhere on the graph where the graph intersects it more than once



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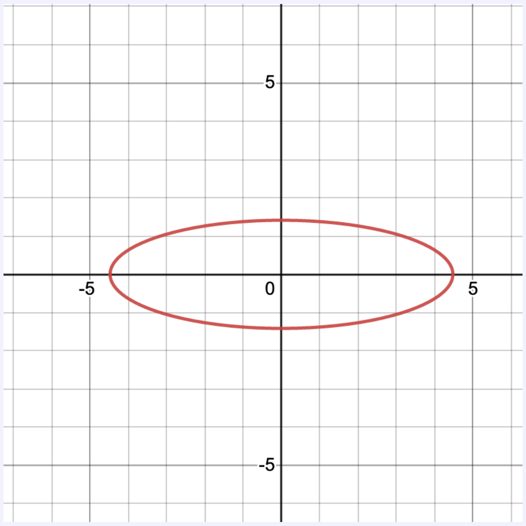
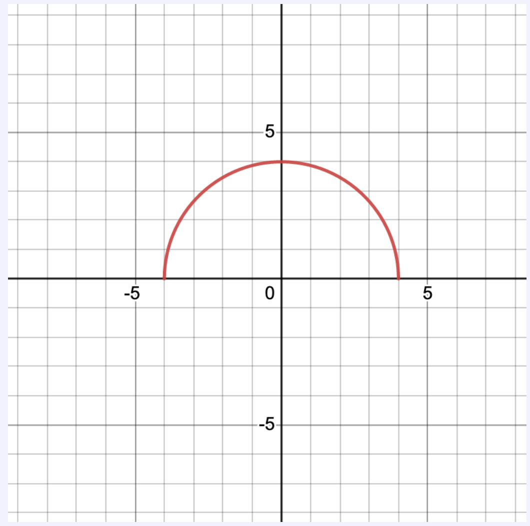
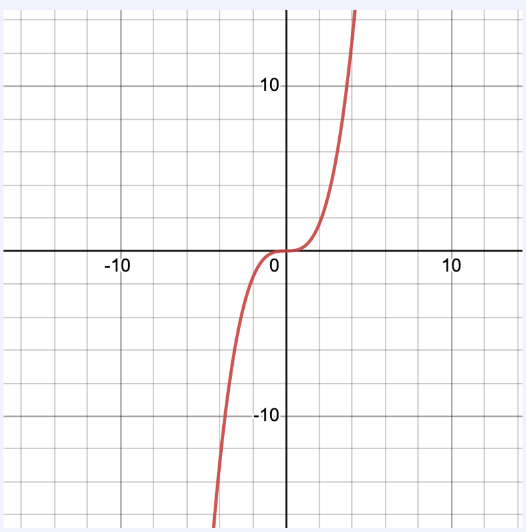
Exercise 7

Determine whether the following graphs are functions or not using the vertical line test.

a) 

b) 

c) 

d)